7.2 Sum and Difference Formulas

1. $\sin\left(\frac{5\pi}{12}\right) = \sin\left(\frac{3\pi}{12}+\frac{2\pi}{12}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6}+\cos\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}\cdot\frac{1}{2}$
$= \frac{1}{4}\left(\sqrt{6}+\sqrt{2}\right)$

2. $\sin\left(\frac{\pi}{12}\right) = \sin\left(\frac{3\pi}{12}-\frac{2\pi}{12}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{6}-\cos\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}-\frac{\sqrt{2}}{2}\cdot\frac{1}{2}$
$= \frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$

3. $\cos\frac{7\pi}{12} = \cos\left(\frac{4\pi}{12}+\frac{3\pi}{12}\right) = \cos\frac{\pi}{3}\cos\frac{\pi}{4}-\sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$
$= \frac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$

4. $\tan\frac{7\pi}{12} = \tan\left(\frac{3\pi}{12}+\frac{4\pi}{12}\right) = \frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{3}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{3}} = \frac{1+\sqrt{3}}{1-1\cdot\sqrt{3}} = \frac{1+\sqrt{3}}{1-\sqrt{3}}\cdot\frac{1+\sqrt{3}}{1+\sqrt{3}}$
$= \frac{1+2\sqrt{3}+3}{1-3} = \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$

5. $\cos 165^\circ = \cos(120^\circ + 45^\circ) = \cos 120^\circ \cos 45^\circ - \sin 120^\circ \sin 45^\circ$
$= -\frac{1}{2}\cdot\frac{\sqrt{2}}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} = -\frac{1}{4}\left(\sqrt{2}+\sqrt{6}\right)$

6. $\sin 105° = \sin(60° + 45°) = \sin 60° \cos 45° + \cos 60° \sin 45°$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{1}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)$$

7. $\tan 15° = \tan(45° - 30°) = \dfrac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°} = \dfrac{1 - \frac{\sqrt{3}}{3}}{1 + 1 \cdot \frac{\sqrt{3}}{3}} = \dfrac{\frac{3-\sqrt{3}}{3}}{\frac{3+\sqrt{3}}{3}}$

$$= \frac{3-\sqrt{3}}{3+\sqrt{3}} \cdot \frac{3-\sqrt{3}}{3-\sqrt{3}} = \frac{9 - 6\sqrt{3} + 3}{9-3} = \frac{12 - 6\sqrt{3}}{6} = \frac{6\left(2 - \sqrt{3}\right)}{6} = 2 - \sqrt{3}$$

8. $\tan 195° = \tan(135° + 60°) = \dfrac{\tan 135° + \tan 60°}{1 - \tan 135° \tan 60°} = \dfrac{-1 + \sqrt{3}}{1 - (-1) \cdot \sqrt{3}} = \dfrac{-1 + \sqrt{3}}{1 + \sqrt{3}} \cdot \dfrac{1 - \sqrt{3}}{1 - \sqrt{3}}$

$$= \frac{-1 + 2\sqrt{3} - 3}{1-3} = \frac{-4 + 2\sqrt{3}}{-2} = 2 - \sqrt{3}$$

9. $\sin \dfrac{17\pi}{12} = \sin\left(\dfrac{15\pi}{12} + \dfrac{2\pi}{12}\right) = \sin \dfrac{5\pi}{4} \cos \dfrac{\pi}{6} + \cos \dfrac{5\pi}{4} \sin \dfrac{\pi}{6} = -\dfrac{\sqrt{2}}{2} \cdot \dfrac{\sqrt{3}}{2} + -\dfrac{\sqrt{2}}{2} \cdot \dfrac{1}{2}$

$$= -\frac{1}{4}\left(\sqrt{6} + \sqrt{2}\right)$$

10. $\tan \dfrac{19\pi}{12} = \tan\left(\dfrac{15\pi}{12} + \dfrac{4\pi}{12}\right) = \dfrac{\tan \frac{5\pi}{4} + \tan \frac{\pi}{3}}{1 - \tan \frac{5\pi}{4} \tan \frac{\pi}{3}} = \dfrac{1 + \sqrt{3}}{1 - 1 \cdot \sqrt{3}} = \dfrac{1 + \sqrt{3}}{1 - \sqrt{3}} \cdot \dfrac{1 + \sqrt{3}}{1 + \sqrt{3}}$

$$= \frac{1 + 2\sqrt{3} + 3}{1-3} = \frac{4 + 2\sqrt{3}}{-2} = -2 - \sqrt{3}$$

11. $\sec\left(-\dfrac{\pi}{12}\right) = \dfrac{1}{\cos\left(-\frac{\pi}{12}\right)} = \dfrac{1}{\cos\left(\frac{3\pi}{12} - \frac{4\pi}{12}\right)} = \dfrac{1}{\cos \frac{\pi}{4} \cos \frac{\pi}{3} + \sin \frac{\pi}{4} \sin \frac{\pi}{3}}$

$$= \frac{1}{\frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}} = \frac{1}{\frac{\sqrt{2} + \sqrt{6}}{4}} = \frac{4}{\sqrt{2} + \sqrt{6}} \cdot \frac{\sqrt{2} - \sqrt{6}}{\sqrt{2} - \sqrt{6}}$$

$$= \frac{4\left(\sqrt{2} - \sqrt{6}\right)}{2-6} = \sqrt{6} - \sqrt{2}$$

12. $\cot\left(-\frac{5\pi}{12}\right) = -\cot\frac{5\pi}{12} = \frac{-1}{\tan\frac{5\pi}{12}} = \frac{-1}{\tan\left(\frac{3\pi}{12}+\frac{2\pi}{12}\right)} = \frac{-1}{\frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{6}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{6}}}$

$= -\left(\frac{1-\tan\frac{\pi}{4}\tan\frac{\pi}{6}}{\tan\frac{\pi}{4}+\tan\frac{\pi}{6}}\right) = -\left(\frac{1-1\cdot\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}\right) = \frac{1-\sqrt{3}}{\sqrt{3}+1}\cdot\frac{1-\sqrt{3}}{1-\sqrt{3}}$

$= \frac{1-2\sqrt{3}+3}{1-3} = \frac{4-2\sqrt{3}}{-2} = -2+\sqrt{3}$

13. $\sin 20°\cos 10° + \cos 20°\sin 10° = \sin(20°+10°) = \sin 30° = \frac{1}{2}$

14. $\sin 20°\cos 80° - \cos 20°\sin 80° = \sin(20°-80°) = \sin(-60°) = -\sin 60° = -\frac{\sqrt{3}}{2}$

15. $\cos 70°\cos 20° - \sin 70°\sin 20° = \cos(70°+20°) = \cos 90° = 0$

16. $\cos 40°\cos 10° + \sin 40°\sin 10° = \cos(40°-10°) = \cos 30° = \frac{\sqrt{3}}{2}$

17. $\frac{\tan 20° + \tan 25°}{1-\tan 20°\tan 25°} = \tan(20°+25°) = \tan 45° = 1$

18. $\frac{\tan 40° - \tan 10°}{1+\tan 40°\tan 10°} = \tan(40°-10°) = \tan 30° = \frac{\sqrt{3}}{3}$

19. $\sin\frac{\pi}{12}\cos\frac{7\pi}{12} - \cos\frac{\pi}{12}\sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{12}-\frac{7\pi}{12}\right) = \sin\left(-\frac{\pi}{2}\right) = -1$

20. $\cos\frac{5\pi}{12}\cos\frac{7\pi}{12} - \sin\frac{5\pi}{12}\sin\frac{7\pi}{12} = \cos\left(\frac{5\pi}{12}+\frac{7\pi}{12}\right) = \cos\left(\frac{12\pi}{12}\right) = \cos\pi = -1$

21. $\cos\frac{\pi}{12}\cos\frac{5\pi}{12} + \sin\frac{5\pi}{12}\sin\frac{\pi}{12} = \cos\left(\frac{\pi}{12}-\frac{5\pi}{12}\right) = \cos\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$

22. $\sin\frac{\pi}{18}\cos\frac{5\pi}{18} + \cos\frac{\pi}{18}\sin\frac{5\pi}{18} = \sin\left(\frac{\pi}{18}+\frac{5\pi}{18}\right) = \sin\left(\frac{6\pi}{18}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$